A Rocket Trip on Cosmological Scales

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1 Motivation

Recently, I got a question from a reader with the silly, but profoundly fun question:

> What impact will the expanding universe have on humans exploring space?

On local scales, the impact is completely non-existent, but on cosmological scales, the expanding universe has a huge impact, especially in a dark energy-drive, exponentially expanding universe like ours.

Ignoring non-possibilities like warp engines, and trivial cases (like turning ourselves into light, or accelerating us to near-light speeds instantaneously), I'd like to imagine a case of putting people on a spaceship, accelerating at g such that there's artificial gravity toward the back of the ship, and then decelerating at g so that there's artificial gravity toward the front.

But in an expanding universe, the target galaxy – the place we'd actually like to travel to – isn't a fixed location. And *that* is the piece of added complexity.

My goal is to figure out how long it will take, both in cosmic time, t, and in rocket (proper) time, τ to travel to an arbitrary point in space. Since an accelerating universe has an ultimate particle horizon, these results will naturally asymptote at $1/H_0$ for a pure Dark Energy Universe.

The technical notes below are my solution to this problem. I introduce most terms, but I'm using fairly standard notation for Special Relativity (as can be found, for instance, in Schutz), and for General Relativity.

While my fully worked example, that of a deSitter Universe (pure Dark Energy) is not an exact description of the current universe, it is a very good approximation over long times (since Dark Energy becomes more and more dominant with time), and perfectly adequate locally.

2 Relativistic Rocket Science in a non-Cosmological Frame

2.1 Work and Energy

Consider a rocketship that starts from rest at the present epoch. The rocket is continuously boosted with a force, mg, giving the impression of artificial, earth-normal gravity to the inhabitants inside. After the rocket has traversed a distance, Δr, total work done is:

$$W = mg\Delta r = \Delta E = mc^2 \Delta\gamma$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1}$$

is the usual time-dilation factor of special relativity.

Thus,

$$\Delta\gamma = \frac{g\Delta r}{c^2}$$

from pure energy conservation considerations.

2.2 Natural Units

To make things a bit more readable (and to get rid of the problems of unit conversion), I'll use so-called "Natural Units," most relevantly:

$$c = 1$$

In Natural Units, all speeds are written dimensionlessly as a fraction of the speed of light.

Likewise, 1 light-year=1 year. yields This yields

$$g = 1.03yr^{-1}$$

and

$$H_0 = \frac{h}{(9.78 \times 10^9 yr)}$$

where the best estimates from Planck are $h = 0.68$, a number I will use throughout.

The γ factor also simplifies:

$$\gamma = \frac{1}{\sqrt{1-v^2}} \quad ; \quad v = \sqrt{1 - \frac{1}{\gamma^2}} \tag{2}$$

which yields:

$$d\gamma = v\gamma^3 dv \tag{3}$$

This relation will come in handy in a bit.

2.3 Time and Distance Relations

Let's reconsider our Work-γ relation, written in Natural Units:

$$d\gamma = g\ dr \tag{4}$$

And though it's kind of trivial, for completeness, we need to note:

$$dr = v\ dt \tag{5}$$

Substituting the above equation and equation (3) into equation (4) yields:

$$v\gamma^3 dg = gvdt$$

Inverting, we get:

$$dv = \pm\frac{g}{\gamma^3}dt \tag{6}$$

where I've put in the $\pm$ signal to indicate the possibility of decelerating. Indeed, for a typical problem, we might suppose that a rocket accelerates for a time, t, and then decelerates for an equal time, bring us to rest at a distant planet.

Notice that we're only writing things down in terms of coordinate time for the moment. We'll also want to consider the proper time inside the rocket:

$$d\tau = \frac{dt}{\gamma}$$

using the standard relations.

2.4 A Simple Analytic Form

The position, coordinate time, proper time, and speed of the ship are all straightforward functions of one another. Supposing our destination were $2r$ from earth, how long would it take to get there?

Let's consider only half the trip, a distance r after which we would decelerate.

$$\gamma = gr + 1$$

and so

$$v = \sqrt{1 - \frac{1}{(gr+1)^2}}$$

using the explcit relation from equation (2).

Recalling:

$$dt = \frac{dr}{v}$$

we almost immediately get:

$$t = \sqrt{\frac{r(gr+2)}{g}} \tag{7}$$

which is precisely what you'd expect. The calculation for τ is also straightforward, but a bit uglier and produces the result

$$\tau = \frac{\ln\left(1 + gr + \sqrt{gr(gr+2)}\right)}{g} \tag{8}$$

Both times are for for the first half of the trip (r should equal to half the distance to be traversed), and the total is found by simply doubling.

3 Accelerating Rockets in an Expanding Universe

Now we get to the crux of the matter. We can imagine trying to solve the same problem in an expanding universe. A rocket leaves earth, accelerates at some fixed rate for billions of years, and then decelerates at the same rate, eventually coming to rest. How far has he traveled? How much coordinate time has elapsed? How much proper time has elapsed?

3.1 The Necessary Cosmology and GR

I will assume (basically because it's both apparently true and also because it doesn't matter) that the universe is described by a flat FRW metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{9}$$

where

$$a(t_0) = 1$$

and subscript "0" always refers to the present epoch.

Thus, the separation between two nearby points in space are given by:

$$dr = adx \tag{10}$$

If I wished to travel to a particular galaxy, I would specify its distance by noting its coordinate position (comoving distance) today. Just to be clear, this is labeled as "x."

Further, we define the Hubble function in the usual way:

$$\frac{\dot{a}}{a} = H(a)$$

This is particularly useful, as a ends up being a proxy for time, since the Hubble relation can be inverted as:

$$dt = \frac{da}{aH(a)} \tag{11}$$

and thus, the rocket acceleration term (equation 6) can be rewritten:

$$dv = \pm\frac{g}{a\gamma^3 H(a)}da \tag{12}$$

3.2 Redshifting

But that's not all. In an expanding universe, a particle in motion will be redshifted, slowing it down without the sense of any local deceleration. This can be derived relatively easily, even in the relativistic limit by noting:

$$U \cdot U = -1$$

and also that the metric is not a function of x meaning that U_1 (For instance) is a conserved quantity. Thus, $U^0 = \gamma$ can be computed as:

$$U^0 = \sqrt{1 + \frac{(U_1)^2}{a^2}}$$

or equivalently:

$$U_1 = \frac{av}{\sqrt{1-v^2}}$$

As a side note, this means that U_1/a is simply defined as the momentum per unit mass of the rocket.

Taking the time derivative yields:

$$0 = \frac{\dot{a}v}{\sqrt{1-v^2}} + a\gamma^3\dot{v}$$

or

$$\dot{v} = -v\frac{1}{\gamma^2}\frac{\dot{a}}{a}$$

or

$$dv = -\frac{v}{\gamma^2}\frac{da}{a} \tag{13}$$

as essentially the cosmological friction term.

3.3 Dynamic Equations

Thus, we can figure out the speed (peculiar velocity) of a rocket starting from rest in an expanding universe:

$$dv = \left(\pm\frac{g}{\gamma^3 H(a)} - \frac{v}{\gamma^2}\right)\frac{da}{a} \tag{14}$$

Before worrying about distance traveled or anything else, it would be nice to be able to solve v as a function of a (and consequently of time).

The Friedmann equation naturally gives an analytic form for the Hubble function:

$$H(a) = H_0\sqrt{\frac{\Omega_M}{a^3} + \Omega_\Lambda + \frac{\Omega_K}{a^2}} \tag{15}$$

where, as always, $\Omega_{M,\Lambda,K}$ represent, respectively, the dimensionless mass density, Dark Energy density, and curvature at the present epoch.

Given the complex interplay of v, γ, and a, it's not obvious how to solve the acceleration equation in general. That said, dimensional analysis helps us a lot. To better than 1% accuracy:

$$\frac{g}{H_0} = 10^{10}h^{-1}$$

The first term on the right of equation (14) will dominate whenever:

$$v\gamma < \frac{g}{H(a)}$$

very, very far into the relativistic regime, and as such, $v \simeq 1$ whenever the cosmological redshifting becomes even remotely relevant.

Thus, there is a critical γ-factor:

$$\gamma_c \equiv \frac{g}{H(a)} \tag{16}$$

such that the rocket achieves terminal velocity.

These rocket trips can therefore be put into 3 categories:

1. Trips for which the rocket never reaches terminal velocity.
2. Trips for which the rocket *does* reach terminal velocity.
3. Trips less than a few thousand light-years, for which we can't reasonably assume relativitistic speeds the entire time.

It's the first two cases that concern us since we have an exact solution for the 3rd.

4 Piecemeal Solutions

While numerical solutions to the rocket problem are possible, figuring out at what point in the trip to decelerate, for instance, becomes extremely tough. Instead, it makes sense to break up the trip into 2 or 3 stages: acceleration, terminal velocity (maybe), and deceleration.

4.1 The Acceleration/Deceleration Stages

In the acceleration stage, we basically assume $v \simeq 1$ the entire time for the purpose of timing the trip according to t. We also get to ignore the drag from the accelerating universe. Thus, we have:

$$dv = \frac{g}{\gamma^3 H(a)}\frac{da}{a}$$

or

$$\gamma^3 dv = g\frac{da}{aH(a)}$$

We'll get the same result in the deceleration stage, of course.

This simplifies a bit to:

$$d(v\gamma) = g\frac{da}{aH(a)}$$

or

$$\Delta v\gamma = g\int_1^{a_1} \frac{da}{aH(a)} \tag{17}$$

where the LHS is the reduced momentum, and the integral on the RHS (evaluated at $a_1 = \infty$) is we'd normally call the particle horizon. a_1 represents the expansion factor of the universe when *either* we hit the critical γ_c, or else we turn around and start decelerating.

4.2 Terminal Velocity Stage

In this case, we have a simple relation:

$$\gamma = \gamma_c(a)$$

Deceleration should begin in time to stop the ship at precisely the intended distance.

4.3 Solutions for an Exponential Universe

The one interesting case that allows a simple analytic solution is that where $H = H_0$, an exponential expansion. This is an exact relation for $\Omega_\Lambda = 1$. At present, our universe has $\Omega_\Lambda \simeq 0.7$, but since Dark Energy becomes more and more dominant over time, it is a good solution to describe the overall effect.

4.3.1 Case 1 (terminal velocity is not reached)

Let us imagine a rocket traveling a short enough distance (but still billions of light-years), but which never reaches terminal velocity. In this case, following the acceleration stage:

$$\Delta v\gamma = \frac{g}{H_0}\ln\left(\frac{a_1}{a_0}\right)$$

for the first half of the trip, and

$$\Delta v\gamma = -\frac{g}{H_0}\ln\left(\frac{a_2}{a_1}\right)$$

for the second. Inspection yields:

$$\frac{a_1}{a_0} = \frac{a_2}{a_1}$$

Likewise, we can relate this to the coordinate time for the trip. As we see from equation (11)

$$\int dt = \frac{1}{H_0}\int\frac{da}{a}$$

so we get:

$$\Delta t = \frac{1}{H_0}\ln\left(\frac{a_1}{a_0}\right)$$

and similarly for the second half. Just to be clear, this means:

$$v_{max}\gamma_{max} = g\Delta t$$

which you might have expected.

How much comoving distance is covered during all of this? To simplify matters we will note that $v \simeq 1$ for essentially the entire trip. Thus:

$$a\ dx = vdt$$

inverts to:

$$dx = \frac{dt}{a} = \frac{1}{H_0}\frac{da}{a^2}$$

And thus:

$$x_1 = \frac{1}{H_0}\left(\frac{1}{a_0} - \frac{1}{a_1}\right)$$

for the first leg and similarly for the second. Or:

$$x_{tot} = \frac{1}{H_0}\left(1 - \frac{1}{a_2}\right) \tag{18}$$

or

$$a_2 = \frac{1}{1 - xH_0} \tag{19}$$

We clearly have a problem at $xH_0 = 1$. Of course we do. That is the particle horizon. Note that no such horizon exists in a matter-dominated universe.

Equivalently:

$$a_1 = \sqrt{\frac{1}{1 - xH_0}} \tag{20}$$

The total trip takes:

$$t = -\frac{1}{H_0}\ln(1 - xH_0) \tag{21}$$

in terms of coordinate time. Note that for small xH_0 this is simply $t = x$.

How about in terms of proper time? As always, we have:

$$d\tau = \frac{dt}{\gamma}$$

so

$$d\tau = \frac{1}{H_0}\frac{da}{a^2}\frac{1}{\gamma}$$

At the beginning of the trip (when most of the proper time elapses, because we're moving at much slower than c), we need to consider the full expansion of:

$$v\gamma = \sqrt{\gamma^2 - 1}$$

So, from earlier:

$$\sqrt{\gamma^2 - 1} = \frac{g}{H_0}\ln\left(\frac{a}{a_0}\right)$$

Solving, we get:

$$\gamma = \sqrt{1 + \left(\frac{g}{H_0}\ln(a/a_0)\right)^2}$$

so

$$\tau = \frac{2}{H_0}\int_1^{a_1}\frac{da}{a^2\sqrt{1 + \left(\frac{g}{H_0}\ln(a/a_0)\right)^2}}$$

This, unfortunately, does not have a nice analytic solution. But any approximations need only be relevant up to $a = e$. Note, also, that since most of the interesting integration is done for $(a-1)g/H_0 < 1$, numerical integrations need to be particularly clever about making sure that steps in a are small.

4.3.2 Case 2 (Terminal Velocity is reached)

In this case, we integrate up to $a = e$, and run at terminal velocity until $a = a_2/e$. As in the previous example, the rocket is basically running at c for the entire trip, so the calculation of x and t remain unchanged. Only τ is different from the previous example.

The middle phase has a constant speed and gamma factor, γ_c. The proper time during this period is easy to compute:

$$\begin{aligned}
\tau_{middle} &= \frac{1}{H_0}\int_e^{a_2/e}\frac{da}{a\gamma_c} \\
&= \frac{1}{\gamma_c H_0}\ln\left(\frac{a_2}{e^2}\right) \\
&= \frac{1}{g}\ln\left(\frac{a_2}{e^2}\right)
\end{aligned}$$

Thus, the total proper time for the trip is:

$$\tau = \frac{2}{H_0}\int_1^e \frac{da}{a^2\sqrt{1+\left(\gamma_c\ln(a/a_0)\right)^2}} + \frac{1}{g}\ln\left(\frac{a_2}{e^2}\right) \tag{22}$$

5 Results

5.1 Travel Time

So how long does a trip take?

Some trips are impossible. It's relatively easy to show that the particle horizon is:

$$x_{max} = \frac{1}{H_0}$$

no matter how quickly you accelerate.

The specific details are obviously a function of the acceleration and deceleration, but supposing we do it at earth normal, a trip of any comoving distance is shown in the figure below.

Most curious of all is that pretty much regardless of how distant you're traveling (within the limits), the maximum travel time is about 45 years according to the rocket.

5.2 Fuel

The time to travel to cosmological distances is surprisingly short. However, the amount of fuel used is enormous. Let us suppose that you have a matter-antimatter engine. Starting from rest, if an amount of fuel, dm is converted to energy then:

$$dE = dm\ c^2$$

of energy is created, which, if it's sent out of the back of the ship, carries a momentum of:

$$p_\gamma = -\frac{dE}{c} = -dm\ c$$

By Newton's 3rd law, this gives an impulse to the ship of

$$\Delta p = dmc$$

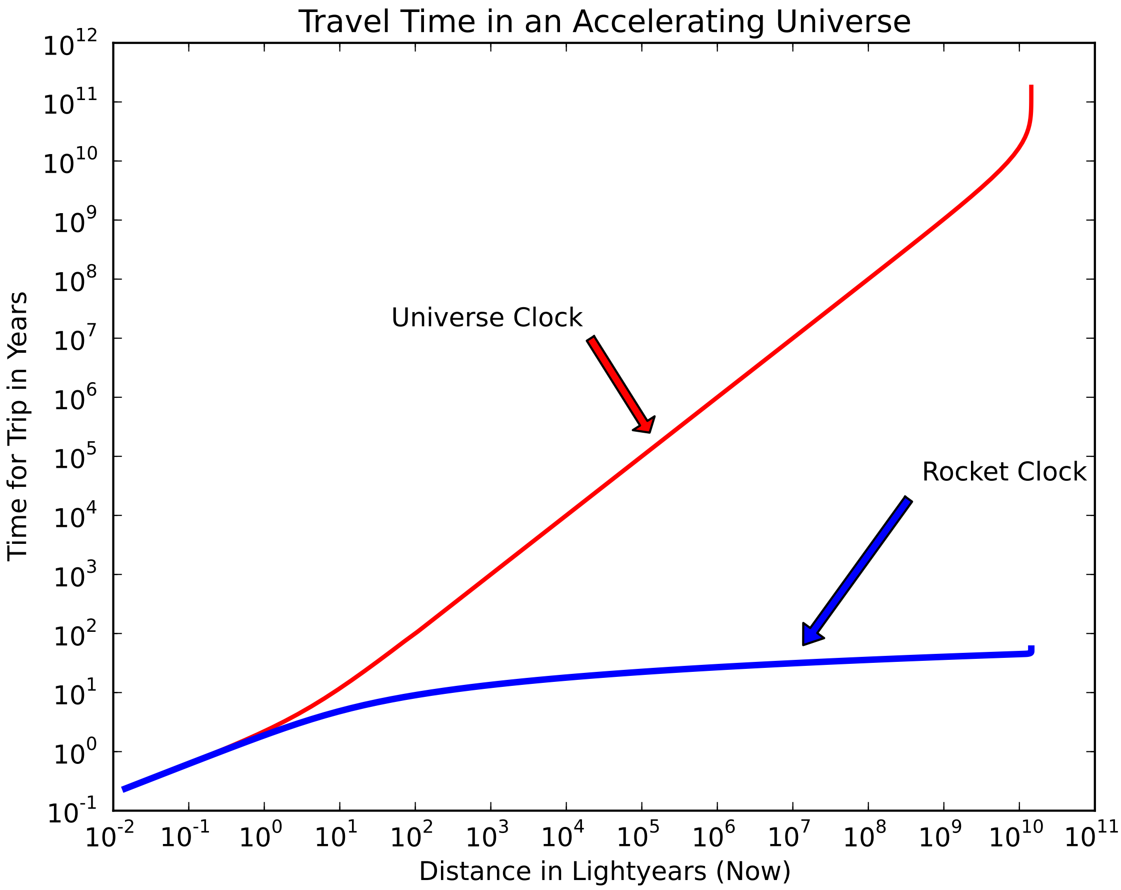

Figure 1: The travel time in both universal time (red) and rocket time (blue) for a trip to a very distant galaxy. Note the limit at the particle horizon.

Thus, according to the ship:

$$M\Delta v = -dMc$$

where I've switched signs because the now dM represents the total loss of payload. Expressing this as a rate per unit time yields:

$$M\frac{\Delta v}{\delta\tau}\Delta\tau = -dMc$$

Note that I've used the proper time, τ. Since in Special Relativity, inertial observers always consider themselves at rest, all that matters is the clock inside the rocketship.

The apparent acceleration (as seen within the ship) is *defined* (within the parameters of our problem) to be:

$$\frac{\Delta v}{\delta\tau} = g$$

and thus:

$$g\ d\tau = -\frac{dM}{M} \tag{23}$$

where I've finally swallowed up $c = 1$. For constant acceleration, this can integrated exactly to:

$$\ln\left(\frac{M_f}{M_i}\right) = g\tau \tag{24}$$

or

$$M_i = M_f \exp(g\tau) \tag{25}$$

To be clear, M_i represents the amount of mass (mostly matter-antimatter fuel) to fly your ship, while M_f represents the mass at the final destination – presumably just the ship, crew, and provisions. For a 45 year trip, this means:

$$M_f = M_i \exp\left(45yr \times 1.03yr^{-1}\right) = M_i \times 1.3 \times 10^{20}$$

Take a mass the size of the moon and use it as fuel, and you can send about 500 kg of material across the cosmos.